## Beta distribution of Second kind

The continuous random variable X which is distributed according to probability law

$$f(x) = \begin{cases} \frac{1}{\beta(a,b)} \cdot \frac{x^{a-1}}{(1+x)^{a+b}} , & 0 < x < \infty , (a,b) > 0 \\ 0 , \text{ otherwise} \end{cases}$$

is known as Beta variate of the second kind with parameters a and b, i.e., $X \sim \beta_2(a,b)$ and its distribution is called Beta distributio of Second kind.

The function f(x) defined above represents a probability function, since total area under the probability curve is unity, i.e.,

$$\int_0^\infty f(x)dx = \int_0^\infty \frac{1}{\beta(a,b)} \cdot \frac{x^{a-1}}{(1+x)^{a+b}} dx$$

$$= \frac{1}{\beta(a,b)} \int_0^\infty \frac{x^{a-1}}{(1+x)^{a+b}} dx$$

$$= \frac{1}{\cancel{\beta(a,b)}} \cdot \cancel{\beta(a,b)} = 1$$

## Properties of Beta distribution of Second kind

### (1) Moments

The $r^{th}$ moment about origin is

$$\mu_r' = E(X^r)$$

$$= \int_0^\infty x^r . f(x) dx$$

$$= \int_0^\infty x^r . \frac{1}{\beta(a,b)} \cdot \frac{x^{a-1}}{(1+x)^{a+b}} dx$$

$$= \frac{1}{\beta(a,b)} \int_0^\infty \frac{x^{r+a-1}}{(1+x)^{a+b}} dx$$

$$= \frac{1}{\beta(a,b)} \int_0^\infty \frac{x^{r+a-1}}{(1+x)^{a+r+b-r}} dx$$

$$= \frac{\beta(a+r, b-r)}{\beta(a,b)}$$

$$= \frac{\Gamma(a+r)\,\Gamma(b-r) / \cancel{\Gamma(a+b)}}{\Gamma(a)\,\Gamma(b) / \cancel{\Gamma(a+b)}}$$

$$= \frac{\Gamma(a+r)\,\Gamma(b-r)}{\Gamma(a)\,\Gamma(b)}$$

In particular,

$$\mu_1' = \frac{\Gamma(a+1)\,\Gamma(b-1)}{\Gamma a\,\Gamma b}$$

$$= \frac{a\,\cancel{\Gamma a}\,\cancel{\Gamma(b-1)}}{\cancel{\Gamma a}\,(b-1)\,\cancel{\Gamma(b-1)}}$$

$$= \frac{a}{(b-1)} = \text{Mean}$$

$$\mu_2' = \frac{\Gamma(a+2)\,\Gamma(b-2)}{\Gamma a\,\Gamma b}$$

$$= \frac{(a+1)\,a\,\cancel{\Gamma a}\,\cancel{\Gamma(b-2)}}{\cancel{\Gamma a}\,(b-1)(b-2)\,\cancel{\Gamma(b-2)}}$$

$$= \frac{a(a+1)}{(b-1)(b-2)}$$

Then

$$\text{Variance} = \mu_2 = \mu_2' - \mu_1'^2$$

$$= \frac{a(a+1)}{(b-1)(b-2)} - \left(\frac{a}{b-1}\right)^2$$

$$= \frac{a}{(b-1)}\left[\frac{(a+1)}{(b-2)} - \frac{a}{(b-1)}\right]$$

$$= \frac{a}{(b-1)}\left[\frac{(a+1)(b-1) - a(b-2)}{(b-2)(b-1)}\right]$$

$$= \frac{a}{(b-1)}\left[\frac{\cancel{ab} - a + b - 1 - \cancel{ab} + 2a}{(b-2)(b-1)}\right]$$

$$= \frac{a(a+b-1)}{(b-1)^2(b-2)}$$

(2) <u>Mode</u> Mode is that value of the variate for which $f(x)$ is maximum, i.e, mode is the solution of

$$f'(x) = 0 \text{ and } f''(x) < 0$$

Therefore

$$f'(x) = \frac{d}{dx} f(x) = 0$$

$$\Rightarrow \frac{d}{dx}\left[\frac{1}{\beta(a,b)} \frac{x^{a-1}}{(1+x)^{a+b}}\right] = 0$$

$$\Rightarrow \frac{1}{\beta(a,b)} \frac{d}{dx}\left[x^{a-1}(1+x)^{-(a+b)}\right] = 0$$

$$\Rightarrow \left[x^{a-1}\{-(a+b)\}(1+x)^{-(a+b)-1} + (1+x)^{-(a+b)}(a-1)x^{a-2}\right] = 0$$

$$\Rightarrow -(a+b)x^{a-1}(1+x)^{-(a+b)-1} + (a-1)x^{a-2}(1+x)^{-(a+b)} = 0$$

$$\Rightarrow \cancel{-}(a+b)x^{a-1}(1+x)^{-(a+b)-1} = \cancel{-}(a-1)x^{a-2}(1+x)^{-(a+b)}$$

$$\Rightarrow \frac{x^{a-1}(1+x)^{-(a+b)-1}}{x^{a-2}(1+x)^{-(a+b)}} = \frac{(a-1)}{(a+b)}$$

$$\Rightarrow x^{\cancel{a}-1-\cancel{a}+2}(1+x)^{-\cancel{(a+b)}-1+\cancel{(a+b)}} = \frac{(a-1)}{(a+b)}$$

$$\Rightarrow x(1+x)^{-1} = \frac{(a-1)}{(a+b)}$$

$$\Rightarrow \frac{x}{1+x} = \frac{(a-1)}{(a+b)}$$

$$\Rightarrow \frac{1+x}{x} = \frac{(a+b)}{(a-1)}$$

$$\Rightarrow \frac{1}{x}+1 = \frac{(a+b)}{(a-1)}$$

$$\Rightarrow \frac{1}{x} = \frac{(a+b)}{(a-1)} - 1$$

$$\Rightarrow \frac{1}{x} = \frac{(a+b)-(a-1)}{(a-1)}$$

$$\Rightarrow \frac{1}{x} = \frac{\cancel{a}+b-\cancel{a}+1}{(a-1)}$$

$$\Rightarrow \frac{1}{x} = \frac{(b+1)}{(a-1)}$$

$$\Rightarrow x = \frac{(a-1)}{(b+1)} = \text{mode value}$$

(3) <u>Harmonic mean</u> The harmonic mean $H$ is given by

$$\frac{1}{H} = E\left(\frac{1}{X}\right)$$

$$= \int_0^\infty \frac{1}{x} f(x)\,dx$$

$$= \int_0^\infty \frac{1}{x} \cdot \frac{1}{\beta(a,b)} \frac{x^{a-1}}{(1+x)^{a+b}}\,dx$$

$$= \frac{1}{\beta(a,b)} \int_0^\infty \frac{x^{a-1-1}}{(1+x)^{a+b}}\,dx$$

$$= \frac{1}{\beta(a,b)} \int_0^\infty \frac{x^{(a-1)-1}}{(1+x)^{a-1+b+1}}\,dx$$

$$= \frac{\beta(a-1,b+1)}{\beta(a,b)} = \frac{\Gamma(a-1)\,\Gamma(b+1)/\cancel{\Gamma(a+b)}}{\Gamma(a)\,\Gamma(b)/\cancel{\Gamma(a+b)}} = \frac{\Gamma(a-1)\,\Gamma(b+1)}{\Gamma(a)\,\Gamma(b)}$$

$$= \frac{\cancel{\Gamma(a-1)}\,\cancel{\Gamma(b)}\,b}{(a-1)\cancel{\Gamma(a-1)}\,\cancel{\Gamma(b)}}$$

$$= \frac{b}{(a-1)}$$

$$\therefore H = \frac{(a-1)}{b}$$